ISRAEL JOURNAL OF MATHEMATICS **131** (2002), 375–379

FIELDS FOR WHICH THE PROJECTIVE SCHUR SUBGROUP IS THE WHOLE BRAUER GROUP

BY

ELI ALJADEFF AND JACK SONN

Department of Mathematics, Technion — Israel Institute of Technology
Haifa 32000, Israel
e-mail: aljadeff@math.technion.ac.il, sonn@math.technion.ac.il

ABSTRACT

Let $Br(K)$ denote the Brauer group of a field K and $PS(K)$ the projective Schur subgroup.

1. Let K be a finitely generated infinite field. Then $PS(K) = Br(K)$ if and only if K is a global field.

2. Let K be a finitely generated infinite field, and let $K((t))$ denote the field of formal power series in t over K. Then $PS(K((t))) = Br(K((t)))$ if and only if $K = \mathbb{Q}$.

1. Introduction

Let K be any field. The **projective Schur (sub)group** $PS(K)$ of a field K is the subgroup of the Brauer group $Br(K)$ generated by (in fact, consisting of) all classes that are represented by a projective Schur algebra A. A finite dimensional K-central simple algebra A is a **projective Schur algebra over** K if the group of units A^* of A contains a subgroup Γ which spans A as a K-vector space and is finite modulo the center, i.e., $K^*\Gamma/K^*$ is a finite group. The notions of projective Schur algebra and the projective Schur group are the projective analogues of Schur algebra and the Schur group of K. These analogues were introduced in 1978 by Lorenz and Opolka [10]. A symbol algebra is a projective Schur algebra in an obvious way (indeed, let $A = (a,b)_n$ be the symbol algebra generated by x and y subject to the relations $x^n = a \in K^*, y^n = b \in K^*, yx = \zeta_n xy$, where $\zeta_n \in K^*$

Received November 4, 2001

is a primitive n-th root of unity. It is easy to see that $\Gamma = < x, y >$ spans A as a K-vector space and $K^*\Gamma/K^* \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. In particular, $K^*\Gamma/K^*$ is a finite group). Invoking the Merkurev–Suslin Theorem, one deduces that if K contains all roots of unity (resp. contains the nth roots of unity), then $PS(K) = Br(K)$ (resp. $PS(K) \supseteq_n Br(K)$) where the subscript n denotes n-torsion. The subgroup $PS(K)$ may be large even if roots of unity are not present in K. Indeed, if K is a number field, then $PS(K) = Br(K)$ as shown in [10]. Here one uses the fact that for K a number field, any element in $Br(K)$ is split by a cyclic extension which is contained in a cyclotomic extension of K. In [2] it was shown that, in general, the projective Schur group $PS(K)$ is properly contained in $Br(K)$. Examples given there were as follows:

1. K a rational function field $k(x)$ over any number field k.

2. For power series fields $K = k((x))$ (over a number field k) the situation depends on the field k. For instance, if k is a number field which is not totally real, then $PS(K) \neq Br(K)$. On the other hand, the Kronecker–Weber Theorem implies that $PS(\mathbb{Q}((x))) = Br(\mathbb{Q}((x))$.

In this paper we prove that, at least for finitely generated fields and formal power series fields over finitely generated fields, the known examples of fields K for which $PS(K) = Br(K)$ are the only ones. Namely we have the following two theorems.

THEOREM 1.1: *Let K be a finitely generated infinite field. Then $PS(K) = Br(K)$ if and only if K is a global field.*

THEOREM 1.2: *Let K be a finitely generated infinite field. Then $PS(K((t))) = Br(K((t)))$ if and only if $K = \mathbb{Q}$.*

2. Proofs

In what follows, F_{ab}, F_{radab}, F_{cyc} and F_{kum} will denote the maximal abelian extension, the maximal radical abelian extension, the maximal cyclotomic extension and the maximal Kummer extension, respectively, of a field F. (A radical extension of F is an extension of F which is generated over F by elements having finite (multiplicative) order modulo F^*.)

Proof of Theorem 1.1: If K is a global field, then $PS(K) = Br(K)$ by [10] (see also [2]; the proof for number fields is also valid for global function fields). For the converse, let K be finitely generated of transcendence degree ≥ 1 over a global field. Then K is a finite extension of a rational function field $k(x_1, \ldots, x_n) = F$, where k is a global field and $n \geq 1$, and without loss of generality k is algebraically closed in K. By [2, Corollary 3.3], if p is any prime for which k does not contain the pth roots of unity, then considering p-primary components, $PS(F)_p \subseteq Br(F_{cyc}/F)_p$, and $Br(F)_p/Br(F_{cyc}/F)_p$ is infinite. Assume in addition that $p \nmid [K_{cyc} : F_{cyc}]$. Take any element $a \in Br(F)_p$ lying outside of $Br(F_{cyc}/F)_p$. Suppose $PS(K)_p = Br(K)_p$. Then $res_{K/F}(a) \in PS(K)_p \subseteq Br(K_{cyc}/K)_p$, hence

$$res_{K_{cyc}/F}(a) = res_{K_{cyc}/K} res_{K/F}(a) = 0 = res_{K_{cyc}/F_{cyc}} res_{F_{cyc}/F}(a).$$

Applying $cor_{K_{cyc}/F_{cyc}}$, we get

$$[K_{cyc} : F_{cyc}] res_{F_{cyc}/F}(a) = 0.$$

Since $res_{F_{cyc}/F}(a)$ is of p-power order $\neq 1$ and $p \nmid [K_{cyc} : F_{cyc}]$, we have a contradiction. ∎

Remark: The same proof works for K finitely generated over any Hilbertian field, provided there are infinitely many primes p such that K does not contain the pth roots of unity.

Problem: Can one characterize the *nonfinitely* generated fields for which $PS(K) = Br(K)$? Theorem 1.2 is a partial result in this direction.

The proof of Theorem 1.2 is based on the following two lemmas.

LEMMA 2.1: *Let K be any field. Then $PS(K((t))) = Br(K((t)))$ if and only if the following two conditions are satisfied:*

(i) $PS(K) = Br(K)$,

(ii) $K_{ab} = K_{radab}$.

Proof: By [7, Theorem 2.5],

$$PS(K((t))) \cong PS(K) \oplus Hom(G(K_{radab}/K), \mathbb{Q}/\mathbb{Z}).$$

Since this isomorphism is the restriction of the isomorphism of Witt's theorem [13, p. 186]

$$Br(K((t))) \cong Br(K) \oplus Hom(G_K, Q/Z),$$

the result follows immediately. ∎

LEMMA 2.2: *Let $K \neq \mathbb{Q}$ be a global field, p a prime number. Then there exists a cyclic extension F/K of degree p not contained in K_{cyc}.*

Proof: Assume first that K is a global function field. For any p, there exists a unique cyclic extension of degree p contained in K_{cyc}. By [9, p. 396] or [11, p. 275] $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is realizable as a Galois group over K. The result follows immediately. Now assume K is a number field different from $\mathbb{Q}$. Let N be its normal closure over $\mathbb{Q}$, $G = G(N/\mathbb{Q})$, $H = G(N/K)$. Consider $\mathbb{Z}/p\mathbb{Z}$ as an H-module with trivial action, and let A be the induced G-module $ind_H^G \mathbb{Z}/p\mathbb{Z}$ [12, p. 28]. We have an embedding problem given by the split exact sequence

$$1 \to A \to \hat{G} \to G \to 1$$

where $\hat{G}$ is the semidirect product of G and A with the given action of G on A. By Scholz's theorem [9, p. 396] or [11, p. 275], this embedding problem has a proper solution with solution field $L \supset N$. In particular, $G(L/\mathbb{Q}) \cong \hat{G}$. By construction, there is a $\mathbb{Z}/p\mathbb{Z}$-extension E of N contained in L such that E/K is Galois with group $H \times \mathbb{Z}/p\mathbb{Z}$, and the normal closure of $E/\mathbb{Q}$ is L. It follows that there exists a $\mathbb{Z}/p\mathbb{Z}$-extension F/K such that $E = FN$.

CLAIM: *F is not contained in K_{cyc}.* If it were, then $FN = E$ would be contained in $N_{cyc} = N\mathbb{Q}_{cyc}$. But G acts trivially on $G(N_{cyc}/N)$, hence every intermediate field between N and N_{cyc} is normal over $\mathbb{Q}$, so in particular E is normal over $\mathbb{Q}$, contradiction, since $E \neq L$ (because of the assumption $K \neq \mathbb{Q}$). ∎

Remark: We thank Moshe Jarden for pointing out that the proof above for number fields holds with $\mathbb{Q}$ replaced by any hilbertian field k, and with $K_{cyc} = K\mathbb{Q}_{cyc} = K\mathbb{Q}_{ab}$ replaced by Kk_{ab}. Thus the lemma holds for any hilbertian field K which is a proper finite extension of a hilbertian field.

Proof of Theorem 1.2: $PS(\mathbb{Q}((t))) = Br(\mathbb{Q}((t))$ was noted already in [2]; it also follows immediately from Lemma 2.1. For the converse, let K be a finitely generated field such that $PS(K((t))) = Br(K((t)))$. By Lemma 2.1 and Theorem 1.1, K is a global field, and $K_{ab} = K_{radab}$. We show that the latter condition holds only for $K = \mathbb{Q}$. There exists a prime p such that K does not contain the pth roots of unity. We claim that if $K \neq \mathbb{Q}$, then there exists a cyclic extension F of K of degree p which is not contained in K_{radab}, which yields the desired contradiction. For this it suffices to show that F is not contained in K_{cyc}, since if it were contained in $K_{radab} = K_{cyc}K_{kum}$ [2], it would be contained in the maximal p-subextension of K_{radab}, which is the composite of the maximal p-subextensions of K_{cyc} and K_{kum} respectively. But since K does not contain the pth roots of unity, the maximal p-subextension of K_{kum} is K itself. The claim now follows from Lemma 2.2, completing the proof of Theorem 1.2. ∎

References

[1] E. Aljadeff and J. Sonn, *Projective Schur algebras have abelian splitting fields,* Journal of Algebra **175** (1995), 179–187.

[2] E. Aljadeff and J. Sonn, *On the projective Schur group of a field,* Journal of Algebra **178** (1995), 530–540.

[3] E. Aljadeff and J. Sonn, *Projective Schur algebras of nilpotent type are Brauer equivalent to radical algebras,* Journal of Algebra **220** (1999), 401–414.

[4] E. Aljadeff and J. Sonn, *Exponent reduction for radical abelian algebras,* Journal of Algebra **223** (2000), 527–534.

[5] E. Aljadeff and J. Sonn, *Exponent reduction for projective Schur algebras,* Journal of Algebra **239** (2001), 356–364.

[6] E. Aljadeff and J. Sonn, *Relative Brauer groups and m-torsion,* Proceedings of the American Mathematical Society **130** (2002), 1333–1337.

[7] E. Aljadeff and J. Sonn, *Projective Schur groups of iterated power series fields,* preprint.

[8] C. Curtis and I. Reiner, *Methods of Representation Theory, Vol. 1,* Wiley, New York, 1981.

[9] M. Fried and M. Jarden, *Field Arithmetic,* Springer-Verlag, Berlin, 1986.

[10] F. Lorenz and H. Opolka, *Einfache Algebren und projektive Darstellungen über Zahlkörpern,* Mathematische Zeitschrift **162** (1978), 175–182.

[11] G. Malle and B. H. Matzat, *Inverse Galois Theory,* Springer-Verlag, Berlin, 1999.

[12] J.-P. Serre, *Linear Representations of Finite Groups,* Springer-Verlag, New York, 1977.

[13] J.-P. Serre, *Local Fields,* Springer-Verlag, New York, 1979.